EXERCISES

ECOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

MATHEMATICS DEPARTMENT

Spectral theory

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1. Operators on Banach and Hilbert spaces

Exercise 1.17. Prove that the inner product of a pre-Hilbert space is continuous in each variable, and that the norm is a continuous function.

Solution. We show that the inner product $\langle\cdot,\cdot\rangle\colon \mathcal{H} \longrightarrow \mathbb{C}$ is continuous in the first variable, and the other proof is similar. Fix then $x, z \in \mathcal{H}$ and $\varepsilon > 0$. Let $\delta := \frac{\varepsilon}{\|z\|}$. If $y \in \mathcal{H}$ is so that $\|x - y\| < \delta$, the Cauchy-Schwarz inequality yields

$$|\langle x, z\rangle - \langle y, z\rangle| = |\langle x - y, z\rangle| \le \|x - y\|\|z\| < \delta\|z\| = \varepsilon$$

and thus $x \longmapsto \langle x, z\rangle$ is continuous, for all $z \in \mathcal{H}$.

For the norm, fix $x \in \mathcal{H}$ and $\varepsilon > 0$. It is enough to take $\delta := \varepsilon$, because if $\|x-y\| < \delta$, then

$$|\|x\| - \|y\|| \le \|x - y\| < \delta = \varepsilon.$$

Exercise 1.27. Let $A \in \mathcal{B}(X)$ and X be a Banach space. Prove that if $\lambda \in \rho(A)$, then $(A - \lambda I)^{-1}$ is defined on the whole space X. Conclude that when X is Banach, $\lambda \in \rho(A)$ if and only if $A - \lambda I$ is a bijective bounded operator.

Solution. By assumption $\lambda \in \rho(A)$ so $\mathrm{Im}(A-\lambda I)$ is dense in X. It suffices to prove it is also closed. To this aim, we appeal the next lemma:

Lemma. Let X be a Banach space and let $A \in \mathcal{B}(X)$. Assume there exists $C > 0$ so that $\|Au\| \ge C\|u\|$ for any $u \in X$. Then $\mathrm{Im}(A)$ is closed in X.

Proof of the lemma. Let $(Au_n)_{n\in\mathbb{N}} \subset \mathrm{Im}(A)$ be a sequence converging to $v \in X$. In particular, $(Au_n)_{n\in\mathbb{N}}$ is Cauchy in X, and since

$$\|u_n - u_m\| \le \frac{1}{C}\|Au_n - A_m\|$$

for any $n, m \in \mathbb{N}$, we see that $(u_n)_{n\in\mathbb{N}}$ is also Cauchy in X. As X is complete, it has a limit, that we call u. From the boundedness of A, it follows that

$$v = \lim_{n\to\infty} Au_n = A(\lim_{n\to\infty} u_n) = Au$$

and $v \in \mathrm{Im}(A)$, which is therefore closed in X. □

We can now finish the exercise. Indeed, as $\lambda \in \rho(A)$, $(A - \lambda I)^{-1}$ is bounded on its domain, whence

$$\|u\| = \|(A - \lambda I)^{-1}(A - \lambda I)u\| \le \|(A - \lambda I)^{-1}\|\|(A - \lambda I)u\|$$

for all $u \in X$. This means $A - \lambda I$ satisfies the hypothesis of the lemma with $C := \frac{1}{\|(A-\lambda I)^{-1}\|}$, and thus $A-\lambda I$ has closed range, which means $(A-\lambda I)^{-1}$ is actually defined on the whole X.

The second claim immediately follows.

Exercise 1.28. Let X be a Banach space, $\lambda \in \mathbb{C}$, and assume there is a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ so that $\|u_n\| = 1$ and $Au_n - \lambda u_n \longrightarrow 0$ as $n \longrightarrow \infty$. Prove that $\lambda \in \sigma(A)$.

Solution. If towards a contradiction we assume that $\lambda \in \rho(A)$, then $A - \lambda I$ is a bijective bounded operator with bounded inverse, and this implies

$$1 = \|u_n\| \le \|(A - \lambda I)^{-1}\| \|Au_n - \lambda u_n\| \longrightarrow 0$$

as $n \to \infty$, which is excluded. Thus $\lambda \in \sigma(A)$.

Exercise 1.38. Show that $(V^\perp)^\perp = \overline{V}$ for any subspace $V \subset \mathcal{H}$. Next, prove that $V^\perp = \overline{V}^\perp$ for any subspace $V \subset \mathcal{H}$.

Solution. First, we note that $(V^\perp)^\perp$ is a closed subset that contains V. This already implies that $(V^\perp)^\perp \supset \overline{V}$. Conversely, let $v \in (V^\perp)^\perp$. According to the orthogonal decomposition theorem, we have a splitting

$$\mathcal{H} = \overline{V} \oplus \overline{V}^\perp$$

and we can write $v = v_1 + v_2$, $v_1 \in \overline{V}$, $v_2 \in \overline{V}^\perp$. Since $V \subset \overline{V}$, it follows that $\overline{V}^\perp \subset V^\perp$, and in particular v is orthogonal to $\overline{V}^\perp$. Hence $\langle v, v_2\rangle = 0$, and we get

$$0 = \langle v, v_2\rangle = \langle v_1 + v_2, v_2\rangle = \langle v_1, v_2\rangle + \langle v_2, v_2\rangle = \|v_2\|^2$$

since $\langle v_1, v_2\rangle = 0$. Thus $v_2 = 0$, meaning that $v = v_1 \in \overline{V}$. This proves $(V^\perp)^\perp \subset \overline{V}$.

For the second equality, we first note that as $V \subset \overline{V}$, we already have $\overline{V}^\perp \subset V^\perp$. For the reverse inclusion, let $v \in V^\perp$, Fix also $x \in \overline{V}$, and choose $(x_n)_{n\in\mathbb{N}} \subset V$ a sequence converging to x. As $x_n \in V$ and $v \in V^\perp$ for all $n \in \mathbb{N}$, it follows that $\langle x_n, v\rangle = 0$ for all $n \in \mathbb{N}$, and the continuity of the inner product now leads to

$$\langle v, x\rangle = \lim_{n\to\infty} \langle x_n, v\rangle = 0.$$

Thus $v \in \overline{V}^\perp$, and this establishes the inclusion $V^\perp \subset \overline{V}^\perp$.

Exercise 1.40. Define the left and the right shift $S, T\colon \ell^2 \longrightarrow \ell^2$ by $(Su)_n := u_{n+1}$ and $(Tu)_1 := 0$, $(Tu)_n := u_{n-1}$, $n \ge 1$.
Show that S and T are bounded, and compute $\|S\|, \|T\|$. Determine S^*, T^*, and find $\sigma_p(S), \sigma_c(S), \sigma_r(S), \sigma_p(T), \sigma_c(T), \sigma_r(T)$.

Solution. First of all if $u \in \ell^2$ we have

$$\|Su\|_2^2 = \sum_{n\ge1} |(Su)_n|^2 = \sum_{n\ge1} |u_{n+1}|^2 = \|u\|_2^2 - |u_1|^2 \le \|u\|_2^2$$

so that $\|S\| \le 1$. Moreover this inequality is an equality if $u_1 = 0$, whence in fact $\|S\| = 1$. The same reasoning gives $\|T\| = 1$. Next, let $u, v \in \ell^2$. Since

$$\langle Su, v\rangle = \sum_{n\ge1} (Su)_n \overline{v_n}$$

$$
\begin{aligned}
&= \sum_{n\geq 1} u_{n+1}\overline{v_n} \\
&= \sum_{n\geq 2} u_n \overline{v_{n-1}} \\
&= \sum_{n\geq 1} u_n \overline{(Tv)_n} \\
&= \langle u, Tv\rangle
\end{aligned}
$$

we see that $S^* = T$. It also follows that $T^* = (S^*)^* = S$. We now turn to finding spectrums of S and T. Let $u \in \ell^2$ and $\lambda \in \mathbb{C}$. Then

$$Su = \lambda u \iff \forall n \geq 1,\ u_{n+1} = \lambda u_n$$

which inductively provides $u_n = \lambda^{n-1}u_1$, $n \in \mathbb{N}$. As $u \in \ell^2$, we must have $\sum_{n\geq 1} |u_n|^2 < \infty$, *i.e.*

$$|u_1|^2 \sum_{n\geq 1} |\lambda|^{2(n-1)} < \infty$$

which implies $|\lambda| < 1$. Conversely, if $|\lambda| < 1$, then λ is an eigenvalue of S, because for instance $u = (1, \lambda, \lambda^2, \dots) \in \ell^2$ verifies $Su = \lambda u$. This establishes that

$$\sigma_p(S) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

is the open unit disk in the complex plane. By a result from the course, $\sigma(S)$ must contain the closure of the open unit disk, which is the closed unit disk. On the other hand, each $\lambda \in \sigma(S)$ satisfies $|\lambda| \leq \|S\| = 1$, so $\sigma(S)$ is contained in the closed unit disk. Hence

$$\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

As also $\sigma_c(S) \cup \sigma_r(S) = \sigma(S) \setminus \sigma_p(S)$, we deduce that

$$\sigma_c(S) \cup \sigma_r(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \mathbb{S}^1$$

is the boundary of the unit disk. Now let's turn to $\sigma_p(S^*) = \sigma_p(T)$. For $\lambda \neq 0$, the equation $Tu = \lambda u$ is equivalent to $0 = \lambda u_1$ and $u_{n-1} = \lambda u_n$ for all $n \geq 2$. Hence $u_1 = 0$, and this in turn implies $u_2 = 0$, $u_3 = 0$ and in fact $u_n = 0$ for all $n \geq 1$. Thus if $\lambda \neq 0$, it is not an eigenvalue. Likewise, $\lambda = 0$ cannot be an eigenvalue, and we have then $\sigma_p(T) = \emptyset$. This allows us to determine the residual spectrum of S, because

$$
\begin{aligned}
\sigma_r(S) &= \{\lambda \in \mathbb{C} : \lambda \notin \sigma_p(S), \overline{\lambda} \in \sigma_p(S^*)\} \\
&= \{\lambda \in \mathbb{C} : \lambda \notin \sigma_p(S), \overline{\lambda} \in \sigma_p(T)\}.
\end{aligned}
$$

As $\sigma_p(T)$ is empty, it follows that $\sigma_r(S) = \emptyset$ as well. Henceforth it appears that

$$\sigma_c(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \mathbb{S}^1.$$

Likewise, $\sigma_r(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma_c(T) = \mathbb{S}^1$.

Exercise 1.42. Let $\mathcal{H}$ be a complex Hilbert space, and $A \in \mathcal{B}(\mathcal{H})$. Prove that if $\langle Au, u\rangle = 0$ for all $u \in \mathcal{H}$ then $A = 0$.

Solution. Let $v, w \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Then, from the assumption

$$0 = \langle A(v + \lambda w), v + \lambda w\rangle = \lambda\langle Aw, v\rangle + \overline{\lambda}\langle Av, w\rangle.$$

For the special value $\lambda = 1$ we get $\langle Aw, v\rangle = -\langle Av, w\rangle$ and for the special value $\lambda = i$, we end it up with $\langle Aw, v\rangle = \langle Av, w\rangle$. Together these two conditions implies

$$\langle Av, w\rangle = 0$$

for all $v, w \in \mathcal{H}$. In particular fixing $v \in \mathcal{H}$ and setting $w := Av$ provides $\|Av\|^2 = \langle Av, Av\rangle = 0$, whence $Av = 0$ for all $v \in \mathcal{H}$. Thus $A = 0$.

An another way of proceeding is to use the assumption and the polarization identity given in Exercise 1.45. This immediately yields to $\langle Au, v\rangle = 0$ for all $u, v \in \mathcal{H}$, and we conclude as above.

Lastly, note that it is essential for $\mathcal{H}$ to be a *complex* Hilbert space. For a counter example in the real case one can consider $\mathcal{H} = \mathbb{R}^2$ and A the rotation by $\frac{\pi}{2}$. $A \neq 0$, but clearly $\langle Au, u\rangle = 0$ for all $u \in \mathbb{R}^2$.

Exercise 1.45. Prove that if $\mathcal{H}$ is a pre-Hilbert space, then

$$\langle u, v\rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2)$$

for any $u, v \in \mathcal{H}$. Next prove that if A is an operator on a Hilbert space $\mathcal{H}$, then

$$\langle Au, v\rangle = \frac{1}{4}(\langle A(u+v), u+v\rangle - \langle A(u-v), u-v\rangle + i\langle A(u+iv), u+iv\rangle - i\langle A(u-iv), u-iv\rangle)$$

then for any $u, v \in \mathcal{H}$.

Solution. Expand directly the right-hand side of both equalities.

Exercise 1.46. Show that the subspace $\mathcal{S}(\mathcal{H})$ of symmetric operators is closed in $\mathcal{B}(\mathcal{H})$. Show also that if $S, B \in \mathcal{S}(\mathcal{H})$, then $SB \in \mathcal{S}(\mathcal{H})$ if and only if $SB = BS$.

Solution. Take a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathcal{H})$, and suppose $A_n \longrightarrow A$ as $n \to \infty$. Then

$$\begin{aligned}\|A - A^*\| &= \|A - A_n + A_n - A^*\| \\ &\leq \|A - A_n\| + \|(A_n - A)^*\| \\ &= 2\|A - A_n\|\end{aligned}$$

since $A_n^* = A_n$ and since $\|S^*\| = \|S\|$ for every bounded operator S. As $\|A - A_n\| \longrightarrow 0$ as $n \to \infty$, we obtain $A = A^*$, and A is symmetric. This shows that $\mathcal{S}(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{H})$.

For the next claim, let S, B be two symmetric operators. If SB is also symmetric then $SB = (SB)^* = B^*S^* = BS$ so B commutes with S. Conversely, if $SB = BS$, we get that

$$SB = BS = B^*S^* = (SB)^*$$

whence SB is symmetric, as wanted.

Exercise 1.48. Let $A \in \mathcal{B}(\mathcal{H})$ be normal, *i.e.* $AA^* = A^*A$. Prove that A is invertible and has bounded inverse if and only if there exists a constant $C > 0$ so that $\|Au\| \geq C\|u\|$ for all $u \in \mathcal{H}$.
Hint : Prove first that A is normal if and only if $\|Au\| = \|A^*u\|$ for any $u \in \mathcal{H}$. Deduce that a normal operator is injective if and only if it has dense range.

Solution. We start by proving the following result:

An operator $A \in \mathcal{B}(\mathcal{H})$ is invertible if and only if it has dense range and there exists $C > 0$ so that $\|Au\| \geq C\|u\|$ for all $u \in \mathcal{H}$.

Proof. $\Longrightarrow$: If $A \in \mathcal{B}(\mathcal{H})$ is invertible, it has dense range, and

$$\|u\| = \|A^{-1}Au\| \leq \|A^{-1}\|\|Au\|$$

for any $u \in \mathcal{H}$, so we set $C := \frac{1}{\|A^{-1}\|}$ and we have the second condition.
$\Longleftarrow$: Conversely, if there is $C > 0$ so that $\|Au\| \geq C\|u\|$ for all $u \in \mathcal{H}$, then A has closed range, as seen in Exercise 1.27. By assumption, $\mathrm{Im}(A)$ is also dense, so it follows that $\mathrm{Im}(A) = \mathcal{H}$, and A is onto. Note lastly that injectivity is a consequence of the fact that $\|Au\| \geq C\|u\|$, $u \in \mathcal{H}$. Thus A is injective, and then invertible. □

Next, we prove the hint. Observe that

$$\|Au\|^2 - \|A^*u\|^2 = \langle (A^*A - AA^*)u, u\rangle$$

for all $u \in \mathcal{H}$. If A is normal, the right-hand side vanishes whence $\|Au\| = \|A^*u\|$ for all $u \in \mathcal{H}$.

Conversely, if this condition holds for all $u \in \mathcal{H}$, we deduce $\langle (A^*A - AA^*)u, u\rangle = 0$ for all $u \in \mathcal{H}$, and Exercise 1.42 provides $A^*A - AA^* = 0$. Thus A is normal. An immediate consequence of this caracterization is that

$$\mathrm{Ker}(A) = \mathrm{Ker}(A^*)$$

for normal operators. This implies $\mathrm{Ker}(A) = \mathrm{Im}(A)^\perp$, and so $\mathrm{Ker}(A)^\perp = (\mathrm{Im}(A)^\perp)^\perp = \overline{\mathrm{Im}(A)}$ by Exercise 1.38. Appealing the orthogonal decomposition theorem, we get

$$\mathcal{H} = \mathrm{Ker}(A) \oplus \mathrm{Ker}(A)^\perp = \mathrm{Ker}(A) \oplus \overline{\mathrm{Im}(A)}$$

which means exactly that A is injective if and only if it has dense range.

We can now finish the exercise. As already seen, if A is invertible, setting $C := \frac{1}{\|A^{-1}\|}$ guarantees $\|Au\| \geq C\|u\|$ for all $u \in \mathcal{H}$. Conversely this condition ensures injectivity of A, which in turn ensures surjectivity of A by what we just proved. Hence A is invertible, and the boundedness of its inverse is a consequence of the open mapping theorem.

Exercise 1.49. Prove directly that the eigenvalues (if any) of a symmetric operator $A \in \mathcal{B}(\mathcal{H})$ are real.

Solution. If $\lambda \in \mathbb{C}$ is an eigenvalue of A, then there is $u \neq 0$ so that $Au = \lambda u$. We compute

$$\lambda\langle u,u\rangle = \langle \lambda u,u\rangle = \langle Au,u\rangle = \overline{\langle Au,u\rangle} = \overline{\lambda}\langle u,,u\rangle$$

and dividing through by $\langle u,u\rangle = \|u\|^2 \neq 0$ leads $\lambda = \overline{\lambda}$. Thus $\lambda \in \mathbb{R}$.

Exercise 1.57. Check that $\leq$ is a partial order on the class of symmetric operators on $\mathcal{H}$.

Solution. Let $A, B, C \in \mathcal{B}(\mathcal{H})$ be symmetric. As the zero operator is positive, we have $A \leq A$, so $\leq$ is reflexive. If $A \leq B$ and $B \leq C$, then $C - B, B - A$ are positive. It follows that

$$\langle (C-A)u,u\rangle = \langle (C-B)u,u\rangle + \langle (B-A)u,u\rangle \geq 0$$

for all $u \in \mathcal{H}$ so $C - A$ is positive, and $\leq$ is transitive. Lastly, if $A \leq B$ and $B \leq A$, we have $\langle (B-A)u,u\rangle \geq 0$ and $\langle (A-B)u,u\rangle \leq 0$ for all $u \in \mathcal{H}$. This forces $\langle (A-B)u,u\rangle = 0$ for any $u \in \mathcal{H}$, and by Exercise 1.42 this implies $A - B = 0$, whence $A = B$. Thus $\leq$ is antisymmetric, and we are done.

Exercise 1.59. Deduce from the proof of Theorem 1.47 that if A is a positive operator, then $\sigma(A) \subset [0,\infty)$.

Solution. Theorem 1.47 already shows that $\sigma(A) \subset \mathbb{R}$ if A is symmetric. It thus remain to exclude negative numbers from the spectrum. Let then $\lambda < 0$. As in the proof of 1.47, one has

$$\|(A-\lambda I)u\|^2 = \|Au\|^2 - 2\lambda\langle Au,u\rangle + \lambda^2\|u\|^2$$

for all $u \in \mathcal{H}$. As now A is positive, $\langle Au,u\rangle \geq 0$, so $-2\lambda\langle Au,u\rangle \geq 0$ for all $u \in \mathcal{H}$. We then deduce that $\|(A-\lambda I)u\|^2 \geq \lambda^2\|u\|^2$ for all $u \in \mathcal{H}$, and thus $\lambda \in \rho(A)$. It follows that $\sigma(A) \subset [0,\infty)$.

Exercise 1.64. Show that the above result is false if P and Q do not commute.

Solution. Consider two operators P and Q on $\mathbb{C}^2$ given by

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

These two operators are positive, but their product $PQ = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is not positive, as it is even not self-adjoint.

Exercise 1.67. Prove that an operator $P \in \mathcal{B}(\mathcal{H})$ is positive if and only if there exists $A \in \mathcal{B}(\mathcal{H})$ so that $P = A^*A$.

Solution. If $P \in \mathcal{B}(\mathcal{H})$ is positive, it suffices to set $A := \sqrt{P}$ to get $A^*A = P$. Conversely, if $P = A^*A$, then

$$\langle Pu, u\rangle = \langle A^*Au, u\rangle = \langle Au, AU\rangle = \|Au\|^2 \geq 0$$

for all $u \in \mathcal{H}$, whence P is positive.

Exercise 1.68. Let $A\colon \ell^2 \longrightarrow \ell^2$ be the *double right shift*, defined by $(Au)_1 = (Au)_2 := 0$ and $(Au)_n := u_{n-2}$, $n \geq 3$.
Show that A is bounded and compute $\|A\|$. Determine A^*, and find $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$. Is A positive? Find $B\colon \ell^2 \longrightarrow \ell^2$ so that $A = B^2$. What can we conclude?

Solution. First of all, we have

$$\|Au\|_2^2 = \sum_{n\geq 1} |(Au)_n|^2 = \sum_{n\geq 3} |u_{n-2}|^2 = \|u\|_2^2$$

for all $u \in \ell^2$, so A is bounded and $\|A\| = 1$. It is easy to check that the adjoint of A is the *double left shift* $A^*\colon \ell^2 \longrightarrow \ell^2$, defined by $(Au)_n := u_{n+2}$, $n \geq 1$, for any $u \in \ell^2$. The same reasoning as in Exercise 1.40 shows that

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$
$$\sigma_c(A) = \mathbb{S}^1,\ \sigma_r(A) = \emptyset$$

and hence $\sigma(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. To conclude, A is not positive, as it is not self-adjoint, but $A = B^2$ where $B\colon \ell^2 \longrightarrow \ell^2$ is the right shift. This shows that being positive is only a sufficient condition to have a square root, and is not necessary.

2. Spectral theorem I

Exercise 2.6. Let $S \in \mathcal{B}(\mathcal{H})$ be symmetric, and $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a corresponding spectral family. Prove that for any $P \in \mathbb{R}[X]$ and $u, v \in \mathcal{H}$, we have

$$\langle P(S)u, v\rangle = \int_m^{M+\varepsilon} P(\lambda)\ \mathrm{d}\langle E_\lambda u, v\rangle$$

where the right-hand side is the Riemann-Stieltjes integral of P with respect to $\phi(\lambda) := \langle E_\lambda u, v\rangle$.

Solution. It is enough to prove the identity in the case $v = u$, and the general case follows from the polarization identity (Exercise 1.45).

Let $u \in \mathcal{H}$. First observe that the right hand side is well-defined, since $\lambda \longmapsto \langle E_\lambda u, u\rangle$ is of bounded variations. Indeed, if m, M are the lower and upper bounds of S and if

$$m = \lambda_0 < \lambda_1 < \cdots < \lambda_n = M + \varepsilon$$

is an arbitrary partition of $[m, M+\varepsilon]$, then

$$\begin{aligned}\sum_{k=1}^{n} |\langle E_{\lambda_k}u, u\rangle - \langle E_{\lambda_{k-1}}u, u\rangle| &= \sum_{k=1}^{n} \langle E_{\lambda_k}u, u\rangle - \langle E_{\lambda_{k-1}}u, u\rangle \\ &= \left\langle \sum_{k=1}^{n} (E_{\lambda_k} - E_{\lambda_{k-1}})u, u \right\rangle \\ &= \langle (E_{M+\varepsilon} - E_m)u, u\rangle \\ &= \|u\|^2\end{aligned}$$

since $(E_\lambda)_{\lambda\in\mathbb{R}}$ is increasing and $E_{M+\varepsilon} = \mathrm{Id}_{\mathcal{H}}$, $E_m = 0$. Now, fix a sequence of partitions $(\Pi_l)_{l\in\mathbb{N}}$ with $|\Pi_l| \longrightarrow 0$ as $l \longrightarrow 0$. Write explicitly

$$m = \lambda_0^l < \lambda_1^l < \cdots < \lambda_{n_l}^l = M + \varepsilon$$

the partition Π_l, to get

$$\begin{aligned}\langle P(S)u, u\rangle &= \left\langle \lim_{l\to\infty} \sum_{k=1}^{n_l} P(\lambda_k^l)(E_{\lambda_k^l} - E_{\lambda_{k-1}^l})u, u \right\rangle \\ &= \lim_{l\to\infty} \sum_{k=1}^{n_l} P(\lambda_k^l)\langle (E_{\lambda_k^l} - E_{\lambda_{k-1}^l})u, u\rangle \\ &= \lim_{l\to\infty} \sum_{k=1}^{n_l} P(\lambda_k^l)(\langle E_{\lambda_k^l}u, u\rangle - \langle E_{\lambda_{k-1}^l}u, u\rangle) \\ &= \int_m^{M+\varepsilon} P(\lambda)\, \mathrm{d}\langle E_\lambda u, u\rangle\end{aligned}$$

as claimed.

3. The spectral theorem for self-adjoint operators

Exercise 3.9. Let $\phi \in L^\infty(\mathbb{R})$ and consider the multiplication operator $T\colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ defined as $(Tu)(x) := \phi(x)u(x)$, $x \in \mathbb{R}$. Show that T is bounded and compute its norm. Find T^*, and determine under which condition T is symmetric.

Now, suppose that $\lim_{x\to+\infty} |\phi(x)| = +\infty$. Show that T is unbounded, and find its domain. Find T^*.

Solution. If $\phi \in L^\infty(\mathbb{R})$, we get that

$$\|Tu\|_2^2 = \int_R |\phi(x)|^2 |u(x)|^2 \,\mathrm{d}x \leq \|\phi\|_\infty^2 \|u\|_2^2$$

for all $u \in L^2(\mathbb{R})$, whence T is bounded and $\|T\| \le \|\phi\|_\infty$. Now let $\varepsilon > 0$. By definition of $\|\phi\|_\infty$, we may find a subset $E \subset \mathbb{R}$ of Lebesgue measure $|E| > 0$ so that $|\phi(x)| \ge \|\phi\|_\infty - \varepsilon$. Set then $u := \frac{1}{\sqrt{|E|}}\mathbf{1}_E$, and observe that

$$\|Tu\|_2^2 = \int_{\mathbb{R}} |\phi(x)|^2|u(x)|^2 \, \mathrm{d}x \ge (\|\phi\|_\infty - \varepsilon)^2 = (\|\phi\|_\infty - \varepsilon)^2\|u\|_2^2$$

which in turn implies $\|Tu\| \ge \|\phi\|_\infty - \varepsilon$. As $\varepsilon > 0$ is arbitrary, we get $\|T\| \ge \|\phi\|_\infty$, and we conclude that $\|T\| = \|\phi\|_\infty$. To continue, the computation

$$\langle Tu, v\rangle = \int_R \phi(x)u(x)\overline{v(x)} \, \mathrm{d}x = \int_R u(x)\overline{\overline{\phi(x)}v(x)} \, \mathrm{d}x$$

valid for all $u, v \in L^2(\mathbb{R})$, shows that the adjoint of T is given by $T^*\colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, $(T^*u)(x) := \overline{\phi(x)}u(x)$. Then, it follows that T is symmetric if and only if $\overline{\phi(x)} = \phi(x)$ for all $x \in \mathbb{R}$, *i.e.* ϕ is $\mathbb{R}-$valued.

Let us now suppose $\lim_{x \longrightarrow +\infty} |\phi(x)| = +\infty$. The domain of T is then

$$\mathcal{D}_T := \left\{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\phi(x)|^2|u(x)|^2 \, \mathrm{d}x < \infty\right\}.$$

Now let $k \in \mathbb{N}$. By assumption, we can find a subset $E_k \subset \mathbb{R}$ of Lebesgue measure $|E_k| > 0$ so that $|\varphi(x)| \ge k$ for a.e. $x \in E_k$. Let then $u_k := \frac{1}{\sqrt{|E_k|}}\mathbf{1}_{E_k}$ to get

$$\|Tu_k\|_2^2 = \int_{\mathbb{R}} |\phi(x)|^2|u_k|^2 \, \mathrm{d}x \ge k^2 = k^2\|u_k\|_2^2$$

for every $k \in \mathbb{N}$. Hence $\|T\| \ge k$ for all $k \in \mathbb{N}$, and T is unbounded.

We now claim that $\mathcal{D}_{T^*} = \mathcal{D}_T$ and that $T^*v = \overline{\phi}v$ for all $v \in \mathcal{D}_{T^*}$. First of all, if $v \in \mathcal{D}_T$, the mapping $u \longrightarrow \langle Tu, v\rangle$ is bounded on $\mathcal{D}_T$, by the Cauchy-Schwarz inequality. This already proves $v \in \mathcal{D}_{T^*}$, and additionally

$$\langle Tu, v\rangle = \int_{\mathbb{R}} \phi(x)u(x)\overline{v(x)} \, \mathrm{d}x = \int_{\mathbb{R}} u(x)\overline{\overline{\phi(x)}v(x)} \, \mathrm{d}x = \langle u, \overline{\phi}v\rangle$$

whence $T^*v = \overline{\phi}v$ on D_T. It remains to prove $\mathcal{D}_{T^*} \subset \mathcal{D}_T$. Let then $v \in \mathcal{D}_{T^*}$, so that

$$\langle Tu, v\rangle = \langle u, T^*v\rangle$$

for all $u \in \mathcal{D}_T$. This last equality can be written as

$$\int_{\mathbb{R}} u(x)\overline{\overline{\phi(x)}v(x)} \, \mathrm{d}x = \int_{\mathbb{R}} u(x)\overline{T^*v(x)} \, \mathrm{d}x$$

for all $u \in \mathcal{D}_T$. Thus $\langle u, \overline{\phi}v - T^*v\rangle = 0$ for all $u \in \mathcal{D}_T$ and as the latter is dense in $L^2(\mathbb{R})$, it follows that $T^*v = \overline{\phi}v$ and $\mathcal{D}_{T^*} \subset \mathcal{D}_T$.

Exercise 3.11. Show that U, V are both bounded, unitary and satisfy $U^2 = \mathrm{Id}_{\mathcal{H}\oplus\mathcal{H}} = -V^2$. Show that U preserves the inner product on $\mathcal{H}\oplus\mathcal{H}$, and that for any subspace $X \subset \mathcal{H}\oplus\mathcal{H}$, we have $V(X^\perp) = V(X)^\perp$.

Solution. That U, V are bounded and satisfy $U^2 = \mathrm{Id}_{\mathcal{H}\oplus\mathcal{H}} = -V^2$ follow from the definition. We prove that V is unitary, and the proof for U is similar. Let $(u,v), (z,w) \in \mathcal{H}\oplus\mathcal{H}$, and write

$$\begin{aligned}\langle V(u,v),(z,w)\rangle &= \langle (v,-u),(z,w)\rangle \\ &= \langle v,z\rangle - \langle u,w\rangle \\ &= \langle (u,v),(-w,z)\rangle\end{aligned}$$

to deduce that $V^*(u,v) := (-v,u)$, $u,v \in \mathcal{H}$, is the adjoint of V. Hence $V^* = -V = V^{-1}$, and V is unitary. To continue, U clearly preserves the inner product of $\mathcal{H}\oplus\mathcal{H}$. Lastly, fix $X \subset \mathcal{H}\oplus\mathcal{H}$. First let $(u,v) \in V(X^\perp)$, and write $(u,v) = V(z,w) = (w,-z)$ for some $(z,w) \in X^\perp$. Now if $(s,t) \in X$, then

$$\langle (u,v), V(s,t)\rangle = \langle (w,-z),(t,-s)\rangle = \langle w,t\rangle + \langle z,s\rangle = \langle (z,w),(s,t)\rangle = 0$$

as $(s,t) \in X$ and $(z,w) \in X^\perp$. This proves that $V(X^\perp) \subset V(X)^\perp$, and the reverse inclusion is similar.

Exercise 3.19. Let $T\colon D_T \longrightarrow \mathcal{H}$ be densely defined.
(i) Show that T is closable if and only if T^* is densely defined, and that in this case $\overline{T} = T^{**}$.
(ii) Show that if T is densely defined and closable, then $(\overline{T})^* = T^*$.

Solution. (i) Suppose first that T is closable, *i.e.* there is an operator S with $T \subset S$ and $G_S = \overline{G_T}$. Then $S^* \subset T^*$ and in particular $\mathcal{D}_{S^*} \subset \mathcal{D}_{T^*}$. Now S is densely defined (because its domain contains $\mathcal{D}_T$ which is already dense by assumption) and closed, so Theorem 3.13 ensures that $\mathcal{D}_{S^*}$ is dense, and thus $\mathcal{D}_{T^*}$ is dense as well. Hence T^* is densely defined.

Conversely, suppose T^* is densely defined. This guarantees that $T^{**} := (T^*)^*$ exists. Now using Lemma 3.12 we compute that

$$G_{T^{**}} = G_{(T^*)^*} = V(\overline{G_{T^*}})^\perp = V(G_{T^*})^\perp = V(G_{T^*}^\perp) = V(V(\overline{G_T})) = -\overline{G_T} = \overline{G_T}$$

using that the graph of T^* is closed (by Proposition 3.8), that V preserves orthogonality, that $V^2 = -\mathrm{Id}_{\mathcal{H}\oplus\mathcal{H}}$ (Exercise 3.11) and that $\overline{G_T}$ is a subspace. Hence T is closable and $\overline{T} = T^{**}$.

(ii) We directly compute that

$$G_{(\overline{T})^*} = V(\overline{G_{\overline{T}}})^\perp = V(G_{\overline{T}})^\perp = V(\overline{G_T})^\perp = G_{T^*}$$

using Lemma 3.12 and that $\overline{T}$ is closed. Hence $(\overline{T})^* = T^*$ as claimed.

Exercise 3.24. Let $\mathcal{H} = L^2(\mathbb{R})$, and $H\colon \mathcal{D}_H \longrightarrow \mathcal{H}$, $\mathcal{D}_H := C_0^\infty(\mathbb{R})$, $H := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$.
(i) Prove that H is symmetric.
(ii) Prove that $H^* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on the domain

$$\mathcal{D}_{H^*} = \{v \in \mathcal{H} : v \in C^1(\mathbb{R}),\ v' \in AC[a,b] \text{ for any } -\infty < a < b < +\infty,\ v'' \in L^2(\mathbb{R})\}.$$

Hint: To prove the inclusion of $\mathcal{D}_{H^*}$ into the right-hand side, think to Du-Bois Reymond's lemma.
(iii) Is H self-adjoint? essentially self-adjoint?

Solution. (i) $\mathcal{D}_{\mathcal{H}}$ is dense in $\mathcal{H} = L^2(\mathbb{R})$, and for all $u, v \in \mathcal{D}_H$ one has

$$\langle Hu, v\rangle = \int_{\mathbb{R}} -u''(x)\overline{v(x)}\,\mathrm{d}x = \int_{\mathbb{R}} u(x)\overline{-v''(x)}\,\mathrm{d}x = \langle u, Hv\rangle$$

integrating by parts twice and using that u, v vanish at infinity. By Lemma 3.16, H is symmetric.

(ii) Let

$$\mathcal{D} := \{v \in \mathcal{H} : v \in C^1(\mathbb{R}),\ v' \in AC[a,b] \text{ for any } -\infty < a < b < +\infty,\ v'' \in L^2(\mathbb{R})\}.$$

First, let $v \in \mathcal{D}$. Then

$$\langle Hu, v\rangle = \int_{\mathbb{R}} u(x)\overline{-v''(x)}\,\mathrm{d}x = \langle u, -v''\rangle$$

for any $u \in \mathcal{D}_H$, whence $|\langle Hu, v\rangle| \le \|u\|\|v''\|$ for any $u \in \mathcal{D}_H$ by Cauchy-Schwarz. Thus $v \in \mathcal{D}_{H^*}$ and as

$$\langle u, H^*v\rangle = \langle Hu, v\rangle = \langle u, -v''\rangle$$

for all $u \in \mathcal{D}_H$ which is dense, we must have $H^*v = -v''$ on $\mathcal{D}$. Hence $\mathcal{D} \subset \mathcal{D}_{H^*}$ and $H^* = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $\mathcal{D}$.

Conversely, let $v \in \mathcal{D}_{H^*}$. Then the function

$$\varphi(x) := \int_0^x \int_0^y H^*v(z)\,\mathrm{d}z\mathrm{d}y$$

is in $C^1(\mathbb{R})$, in $AC[0,1]$ and $\varphi'' = H^*v \in L^2(\mathbb{R})$. In other words, $\varphi \in \mathcal{D}$. Moreover, for any $u \in \mathcal{D}_H$ we also have

$$\int_{\mathbb{R}} -u''(x)\overline{v(x)}\,\mathrm{d}x = \langle Hu, v\rangle = \langle u, H^*v\rangle = \langle u, \varphi''\rangle = \int_{\mathbb{R}} u''(x)\overline{\varphi(x)}\,\mathrm{d}x$$

and it follows that

$$\int_{\mathbb{R}} u''(x)\overline{\varphi(x) + v(x)}\,\mathrm{d}x = 0$$

for all $u \in \mathcal{D}_H$. By the Du-Bois Reymond lemma, there exist $c_0, c_1 \in \mathbb{C}$ so that $v(x) = \varphi(x) + c_1x + c_0$, and thus $v \in \mathcal{D}$ as well. Hence, $\mathcal{D}_{H^*} \subset \mathcal{D}$ and we are done.

(iii) H cannot be self-adjoint, as its domain and the domain of its adjoint do not coincide. However, it is in fact essentially self-adjoint. To prove this, we make use of Theorem 3.22, and it is enough to prove that $\mathrm{Im}(H+iI), \mathrm{Im}(H-iI)$ are dense in $\mathcal{H}$. Equivalently, we show that $\mathrm{Ker}(H+iI)$ and $\mathrm{Ker}(H-iI)$ reduce to $\{0\}$. If $u \in \mathrm{Ker}(H+iI)$, then

$$u'' = iu.$$

Solving this differential equation provides two independent solutions

$$u_1(x) = \exp\left(\frac{1+i}{\sqrt{2}}x\right),\ u_1(x) = \exp\left(-\frac{1+i}{\sqrt{2}}x\right).$$

Since neither belong to $L^2(\mathbb{R})$, we conclude indeed that $\mathrm{Ker}(H+iI) = \{0\}$, and similarly for the second kernel. Thus H is essentially self-adjoint.

4. Applications to quantum mechanics

Exercise 4.1. Check that $U \in \mathcal{B}(\mathcal{H})$ is unitary if and only if U is surjective and $\langle Uu, Uv\rangle = \langle u, v\rangle$ for all $u, v \in \mathcal{H}$. Is the surjectivity assumption really necessary? Deduce that a unitary operator has norm 1.

Solution. First, if U is unitary, then in particular $U^*U = \mathrm{Id}_{\mathcal{H}}$ and thus

$$\langle Uu, Uv\rangle - \langle u, v\rangle = \langle U^*Uu, v\rangle - \langle u, v\rangle = \langle (U^*U - \mathrm{Id}_{\mathcal{H}})u, v\rangle = 0$$

for any $u, v \in \mathcal{H}$. We deduce that $\langle Uu, Uv\rangle = \langle u, v\rangle$ for all $u, v \in \mathcal{H}$.

Conversely, note that preserving the inner product forces U to be injective, and thus invertible, and additionally

$$\langle (U^*U - \mathrm{Id})u, v\rangle = \langle Uu, Uv\rangle - \langle u, v\rangle = 0$$

for all $u, v \in \mathcal{H}$, whence $U^*U = \mathrm{Id}_{\mathcal{H}}$ by Exercise 1.42. Next observe that

$$U^*(UU^*)U = (U^*U)(U^*U) = \mathrm{Id}_{\mathcal{H}} = U^*U$$

and multiplying from the left by $(U^*)^{-1}$ and from the right by U^{-1} provides $UU^* = \mathrm{Id}_{\mathcal{H}}$. It follows that U is unitary, and also easily that U has norm 1.

The surjectivity assumption is crucial. Indeed the right shift A on $\ell^2(\mathbb{N})$ preserves the inner product, but is not surjective, as any sequence whose first coordinate is not 0 does not lie in its image. On the other hand, it is not a unitary operator as its adjoint A^* is the left shift and that the composite AA^* is not the identity on $\ell^2(\mathbb{N})$.

Exercise 4.3. Show that a one-parameter unitary group $(U_t)_{t\in\mathbb{R}}$ is strongly continuous if and only if it is weakly continuous.

Solution. Using the continuity of the inner product in the first variable (Exercise 1.17), it follows easily that a strongly continuous one-parameter unitary group is weakly continuous.

Let us show the converse. Suppose $t_n \to t^*$, where $(t_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ and $t^* \in \mathbb{R}$. Let $u \in \mathcal{H}$, and write

$$\|U_{t_n}u - U_{t^*}u\|^2 = \|U_{t_n}u\|^2 + \|U_{t^*}u\|^2 - 2\mathrm{Re}\langle U_{t_n}u, U_{t^*}u\rangle = 2\|u\|^2 - 2\mathrm{Re}\langle U_{t_n}u, U_{t^*}u\rangle$$

using that U_{t^*}, U_{t_n}, $n \in \mathbb{N}$, are unitary. As $t \longmapsto \langle U_t u, U_{t^*}u\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{C}$ by assumption, $\langle U_{t_n}u, U_{t^*}u\rangle$ converges to $\|U_{t^*}u\|^2 = \|u\|^2$ as $n \to \infty$. Henceforth

$$\lim_{n\to\infty} \|U_{t_n}u - U_{t^*}u\|^2 = 2\|u\|^2 - 2\|u\|^2 = 0$$

and thus $U_{t_n}u \to U_{t^*}u$ as $n \to \infty$. This shows that $t \longmapsto U_t u$ is continuous, and thus $(U_t)_{t\in\mathbb{R}}$ is strongly continuous.

Exercise 4.4. For $a \in \mathbb{R}$, let $U_a\colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, $(U_a f)(x) := f(x-a)$. Show that $(U_a)_{a\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group.

Solution. Clearly, $U_0 = \mathrm{Id}_{L^2(\mathbb{R})}$ and if $a, a' \in \mathbb{R}$, $f \in L^2(\mathbb{R})$, then

$$U_a(U_{a'}f)(x) = (U_{a'}f)(x-a) = f(x-a-a') = f(x-(a+a')) = (U_{a+a'}f)(x)$$

for any $x \in \mathbb{R}$, whence $U_aU_{a'} = U_{a+a'}$ for all $a, a' \in \mathbb{R}$. In particular, if $a \in \mathbb{R}$, U_a is invertible (its inverse is U_{-a}) and it is thus enough to prove its preserves the inner product on $L^2(\mathbb{R})$ to prove it is unitary. Let then $f, g \in L^2(\mathbb{R})$, and note that

$$\begin{aligned}\langle U_a f, U_a g\rangle &= \int_{\mathbb{R}} (U_a f)(x)\overline{(U_a g)(x)}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x-a)\overline{g(x-a)}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(y)\overline{g(y)}\,\mathrm{d}y \\ &= \langle f, g\rangle\end{aligned}$$

by a change of variable. Hence $(U_a)_{a\in\mathbb{R}}$ is a one-parameter unitary group.

Let us now check the strong continuity. Let $f \in L^2(\mathbb{R})$, $a \in \mathbb{R}$ and $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ so that $a_n \to a$. Let $\varepsilon > 0$. As continuously differentiable compactly supported functions are dense in $L^2(\mathbb{R})$, we may find $g \in C^1(\mathbb{R})$ supported on a compact set $K \subset \mathbb{R}$ so that

$$\|f - g\|_2^2 < \varepsilon.$$

Now we write

$$\begin{aligned}\|U_{a_n}f - U_a f\|_2^2 &= \int_{\mathbb{R}} |f(x-a_n) - f(x-a)|^2\,\mathrm{d}x \\ &\le \int_{\mathbb{R}} |f(x-a_n) - g(x-a_n)|^2\,\mathrm{d}x + \int_{\mathbb{R}} |g(x-a_n) - g(x-a)|^2\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} |g(x-a) - f(x-a)|^2\,\mathrm{d}x\end{aligned}$$

for all $n \in \mathbb{N}$. The first and last integral are bounded by ε. On the other hand, by the mean value theorem, we may write

$$|g(x-a_n) - g(x-a)|^2 = |g'(c)|^2 |a_n - a|^2$$

for some c between a_n and a, and as $g \in C^1(\mathbb{R})$, its derivative g' is continuous on the compact K, therefore bounded, and there is $C > 0$ so that

$$|g(x-a_n) - g(x-a)|^2 \leq C|a_n - a|^2, \ n \in \mathbb{N}.$$

Thus, the second integral above is bounded by $C|K||a_n - a|^2$, and since $a_n \to a$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|U_{a_n} f - U_a f\|_2^2 \leq 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that $U_{a_n} f \to U_a f$ as $n \to \infty$, and $(U_a)_{a \in \mathbb{R}}$ is strongly continuous.